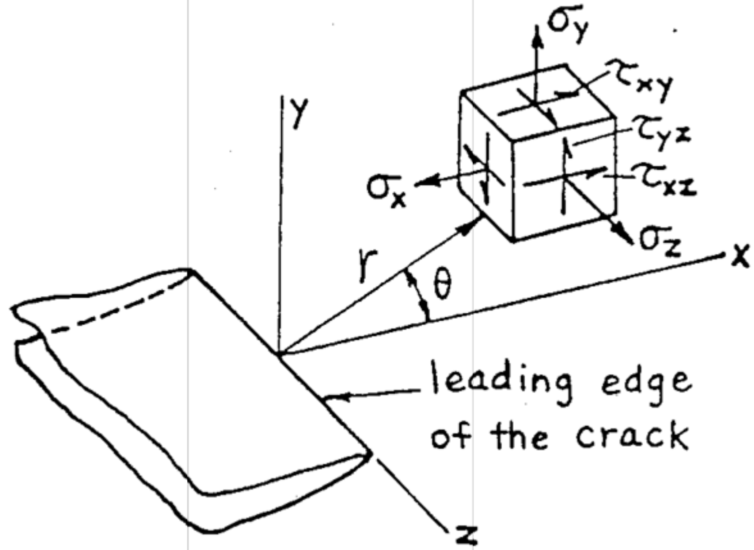
Fundamentals of Fracture Mechanics

- Evolution from the early work of Griffith (1921)
- Fatigue is one of the successful applications of fracture mechanics
- Mechanics for the prediction of crack growth
- Need to define a crack driving force
- Similitude and crack-tip fields
- Three modes of loading - Modes I, II and III

Linear Elastic Fracture Mechanics

- **Isotropic homogeneous elastic materials assumed**
- **Elastic properties: E, ν, $\kappa = 3 - 4\nu$ (plane strain)**
  $\kappa = (3 - \nu)/(1 + \nu)$ **(plane stress)**
  $G = E / 2(1 + \nu)$
- **Stress and displacement fields w.r.t. crack-tip**
  $\sigma_{\alpha\beta} = f(r, \theta, K)$
  $\mu_{\alpha} = f(r, \theta, K)$
  Hence similitude concept



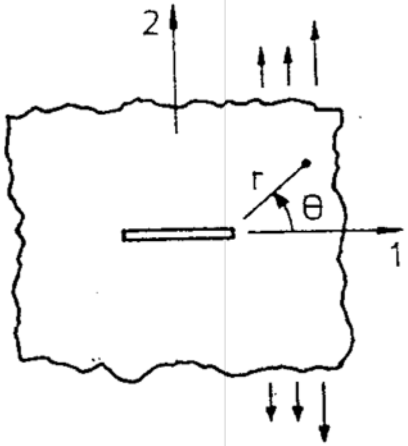
Mode I Stress and Displacement Fields

- **Crack-Tip Stress Fields**

$$\sigma_x = \frac{K_I}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \sigma_{x0} + O\left(r^{1/2}\right)$$

$$\sigma_y = \frac{K_I}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + O\left(r^{1/2}\right)$$

$$\tau_{xy} = \frac{K_I}{(2\pi r)^{1/2}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + O\left(r^{1/2}\right)$$



- **Crack-Tip Displacement Fields (Plane Strain with Higher Order Terms Omitted)**

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right), \tau_{xz} = \tau_{yz} = 0$$

$$\mu = \frac{K_I}{G}\left[r/(2\pi)\right]^{1/2} \cos\frac{\theta}{2}\left[1 - 2\nu + \sin^2\frac{\theta}{2}\right]$$

$$\upsilon = \frac{K_I}{G}\left[r/(2\pi)\right]^{1/2} \sin\frac{\theta}{2}\left[1 - 2\nu - \cos^2\frac{\theta}{2}\right]$$

$$\omega = 0$$

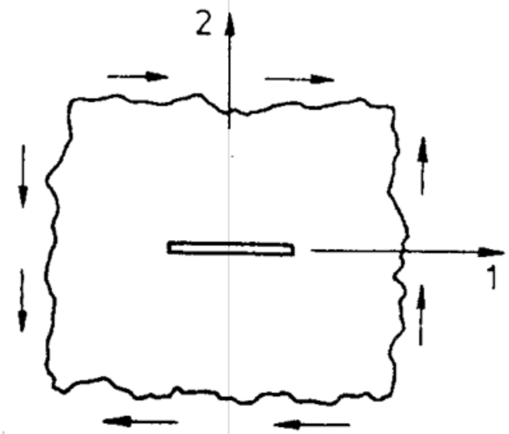
Mode II Stress and Displacement Fields

- **Crack-Tip Stress Fields**

$$\sigma_x = \frac{K_{II}}{(2\pi r)^{1/2}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] + \sigma_{x0} + O\left(r^{1/2}\right)$$

$$\sigma_y = \frac{K_{II}}{(2\pi r)^{1/2}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + O\left(r^{1/2}\right)$$

$$\tau_{xy} = \frac{K_{II}}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + O\left(r^{1/2}\right)$$



- **Crack-Tip Displacement Fields (Plane Strain with Higher Order Terms Omitted)**

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right), \tau_{xz} = \tau_{yz} = 0$$

$$\mu = \frac{K_{II}}{G}\left[r/(2\pi)\right]^{1/2} \sin\frac{\theta}{2}\left[2 - 2\nu + \cos^2\frac{\theta}{2}\right]$$

$$\upsilon = \frac{K_{II}}{G}\left[r/(2\pi)\right]^{1/2} \cos\frac{\theta}{2}\left[-1 + 2\nu - \sin^2\frac{\theta}{2}\right]$$

$$\omega = 0$$

Mode III Stress and Displacement Fields

- **Crack-Tip Stress Fields**

$$\tau_{xz} = \frac{K_{III}}{(2\pi r)^{1/2}} \sin\frac{\theta}{2} + \tau_{xz0} + O\left(r^{1/2}\right)$$

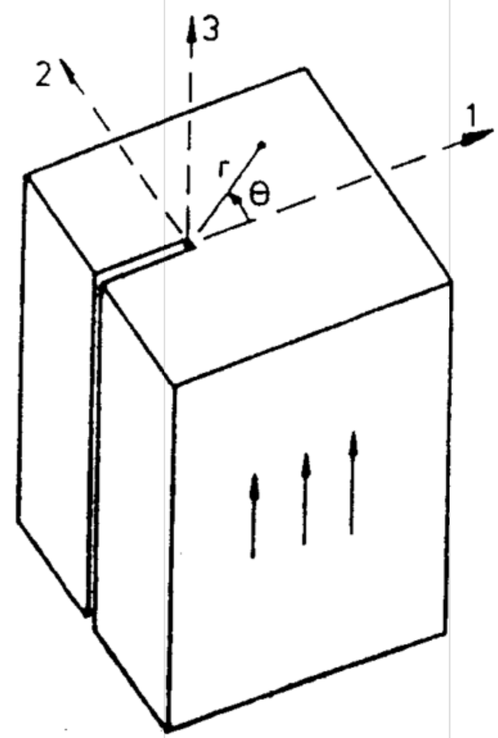
$$\tau_{yz} = \frac{K_{III}}{(2\pi r)^{1/2}} \cos\frac{\theta}{2} + O\left(r^{1/2}\right)$$

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

- **Crack-Tip Displacement Fields (Plane Strain with Higher Order Terms Omitted)**

$$\omega = \frac{K_{III}}{G}\left[(2r)/\pi\right]^{1/2} \sin\frac{\theta}{2}$$

$$\mu = \upsilon = 0$$

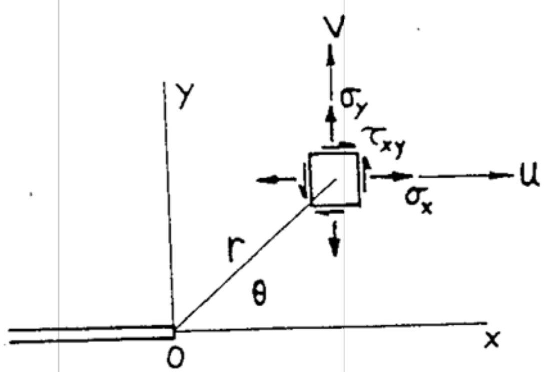


Alternate Expressions Crack-Tip Stress & Displacement Fields (X - Y Components)

Mode I

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \begin{Bmatrix} 1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ 1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{3\theta}{2} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \frac{1}{4} \begin{Bmatrix} 3\cos\frac{\theta}{2} + \cos\frac{5\theta}{2} \\ 5\cos\frac{\theta}{2} - \cos\frac{5\theta}{2} \\ -\sin\frac{\theta}{2} + \sin\frac{5\theta}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{K_I}{G}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{Bmatrix} \left(\beta - \cos^2\frac{\theta}{2}\right) = \frac{K_I}{G}\sqrt{\frac{r}{2\pi}} \frac{1}{4} \begin{Bmatrix} (4\beta - 3)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\ (4\beta - 1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2} \end{Bmatrix}$$



Mode

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{K_{II}}{\sqrt{2\pi r}}\begin{Bmatrix}-\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\ \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\end{Bmatrix}=\frac{K_{II}}{\sqrt{2\pi r}}\frac{1}{4}\begin{Bmatrix}-5\sin\frac{\theta}{2}-\sin\frac{5\theta}{2}\\ -\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}\\ 3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\end{Bmatrix}$$

$$\begin{Bmatrix}u\\ v\end{Bmatrix}=\frac{K_{II}}{G}\sqrt{\frac{r}{2\pi}}\begin{Bmatrix}\sin\frac{\theta}{2}\left(\beta+\cos^2\frac{\theta}{2}\right)\\ -\cos\frac{\theta}{2}\left(\beta-2+\cos^2\frac{\theta}{2}\right)\end{Bmatrix}=\frac{K_{II}}{G}\sqrt{\frac{r}{2\pi}}\frac{1}{4}\begin{Bmatrix}(4\beta+1)\sin\frac{\theta}{2}-\sin\frac{3\theta}{2}\\ -(4\beta-5)\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\end{Bmatrix}$$

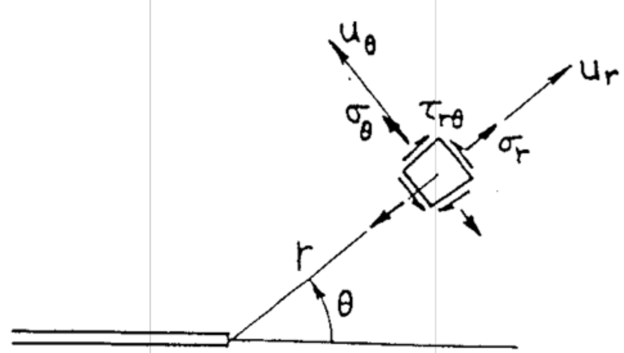
where $\beta(=1/\alpha)=\begin{cases}2(1-\nu)\cdot \text{plane}\cdot\text{strain}\\ 2\left(\frac{1}{1+\nu}\right)\cdot\text{plane}\cdot\text{stress}\end{cases}$

Alternate Expressions For Crack-Tip Stress and Displacement Fields
(r - θ Components)

Mode I

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \begin{Bmatrix} 1+\sin^2\frac{\theta}{2} \\ \cos^2\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \frac{1}{4} \begin{Bmatrix} 5\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\ 3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \\ \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} u_r \\ u_\theta \end{Bmatrix} = \frac{K_I}{G}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{Bmatrix} \left(\beta - \cos^2\frac{\theta}{2}\right) = \frac{K_I}{G}\sqrt{\frac{r}{2\pi}} \frac{1}{4} \begin{Bmatrix} (4\beta-3)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\ -(4\beta-1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \end{Bmatrix}$$



Mode II

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}=\frac{K_{II}}{\sqrt{2\pi r}}\begin{Bmatrix}\sin\frac{\theta}{2}\left(1-3\sin^2\frac{\theta}{2}\right)\\ -\sin\frac{\theta}{2}3\cos^2\frac{\theta}{2}\\ \cos\frac{\theta}{2}\left(1-3\sin^2\frac{\theta}{2}\right)\end{Bmatrix}=\frac{K_{II}}{\sqrt{2\pi r}}\frac{1}{4}\begin{Bmatrix}-5\sin\frac{\theta}{2}+3\sin\frac{3\theta}{2}\\ -3\sin\frac{\theta}{2}-3\sin\frac{3\theta}{2}\\ \cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\end{Bmatrix}$$

$$\begin{Bmatrix}u_r\\ u_\theta\end{Bmatrix}=\frac{K_{II}}{G}\sqrt{\frac{r}{2\pi}}\begin{Bmatrix}-\sin\frac{\theta}{2}\left(\beta-3\cos^2\frac{\theta}{2}\right)\\ -\cos\frac{\theta}{2}\left(\beta+2-3\cos^2\frac{\theta}{2}\right)\end{Bmatrix}=\frac{K_{II}}{G}\sqrt{\frac{r}{2\pi}}\frac{1}{4}\begin{Bmatrix}-(4\beta-3)\sin\frac{\theta}{2}+3\sin\frac{3\theta}{2}\\ -(4\beta-1)\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\end{Bmatrix}$$

where $\beta(=1/\alpha)=\begin{cases}2(1-\nu)\cdot\text{plane}\cdot\text{strain}\\ 2\left(\frac{1}{1+\nu}\right)\cdot\text{plane}\cdot\text{stress}\end{cases}$

Note: For Mode I, $u_r=u, u_\theta=-v$ **(Displacement is in** $\theta/2$ **direction)**

Relationships Between G and K

- **Total energy release rate in general subdivided for each mode**

$$G = G_I + G_{II} + G_{III}$$

where

$$G_I = K_I^2 / E'$$
$$G_{II} = K_{II}^2 / E'$$
$$G_{III} = K_{III}^2 / 2G = \frac{1+\upsilon}{E} K_{III}^2$$

- **Above relationships apply mainly to straight cracks**
- **Different relationships needed for anisotropic bodies (Tada, Paris & Irwin, 1985)**

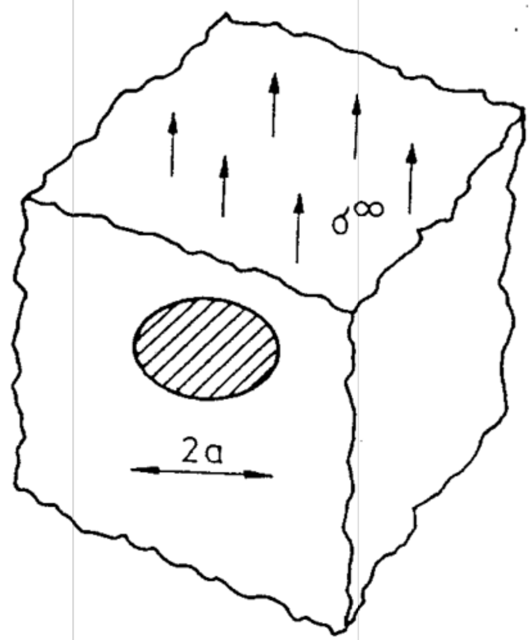
Catalogs/Sources of Stress Intensity Factor Solutions

- Stress Analysis of Cracks handbook (Tada, Paris & Irwin, 1985)
- SURFCAN1 Software (Through-Thickness/Semi-Elliptical Cracks)
- Other Handbooks By
  - -- Sih
  - -- Murakami
  - -- Rooke and Cartwright

Typical Specimen Geometries (Cont.)

Penny shaped crack, Near crack edge fields are identical to Mode I plane strain

$$K_I = \frac{2}{\pi} \sigma^\infty \sqrt{\pi a}$$

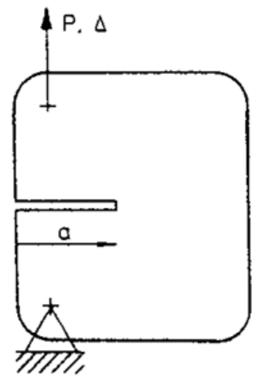


Energy Release Rate and Load Induced Displacement

- **Consider Mode I problem with load P per unit thickness**

PE = potential energy per unit thickness
Δ = load point displacement
G = energy release rate per unit thickness

$$G = -\left(\frac{\partial PE}{\partial a}\right)_P$$



- **For a prescribed P**

$$PE = SE - P\Delta = \frac{1}{2}P\Delta - P\Delta = -P\Delta$$

$$G = \frac{1}{2}\frac{\partial}{\partial a}(P\Delta)\Big|_P = \frac{P}{2}\left(\frac{\partial \Delta}{\partial a}\right)_P$$

Compliance Analysis of Loaded Specimen

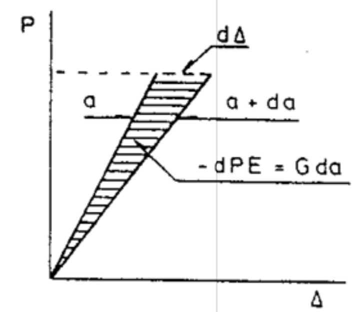
- **Definition of compliance C = Δ/P**

  where C depends only on geometry, E and ν

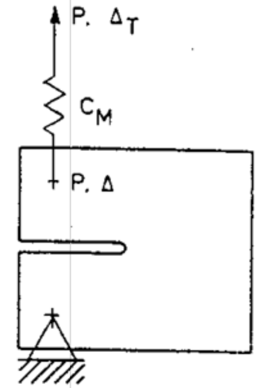
$$\left(\frac{\partial \Delta}{\partial a}\right)_P = P\frac{dc}{da}$$

and $G = \frac{P}{\mathbf{2}}\left(\frac{\partial \Delta}{\partial a}\right)_P$

Hence $G = \frac{P^2}{\mathbf{2}}\frac{dc}{da}$



Effects of Machine Compliance

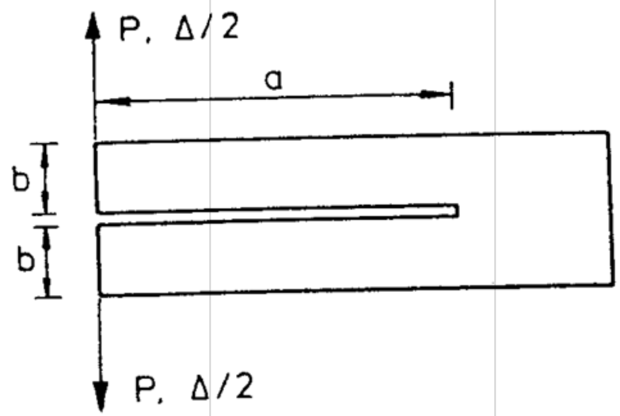


- **C_M = Machine compliance (testing machine)**
- **C_M in series with specimen**
- $\Delta_T = \Delta + C_M P = \Delta + (C_M / C)\Delta$ **= total displacement**
- $PE = SE + \frac{1}{2} C_M P^2 = \frac{1}{2} C^{-1} \Delta^2 + \frac{1}{2} C_M^{-1} (\Delta_T - \Delta)^2$
- $G = -\left(\frac{\partial PE}{\partial a}\right)_{\Delta_T} = -\left[C^{-1}\Delta - C_M^{-1}(\Delta_T - \Delta)\right]\left(\frac{\partial \Delta}{\partial a}\right)_{\Delta_T} + \frac{1}{2} C^{-2} \Delta^2 \frac{dc}{da}$

$$= \frac{1}{2} C^{-2} \Delta^2 \frac{dc}{da} = \frac{1}{2} P^2 \frac{dc}{da}$$

Energy Method For Estimating G and K

- **Double Cantilever Example**



Treat each arm of specimen as cantilever beam

$$\frac{\Delta}{2} = \frac{P_a{}^3}{3EI} = \frac{4P_a{}^3}{Eb^3}$$

Hence

$$C = \frac{8a^3}{Eb^3}$$

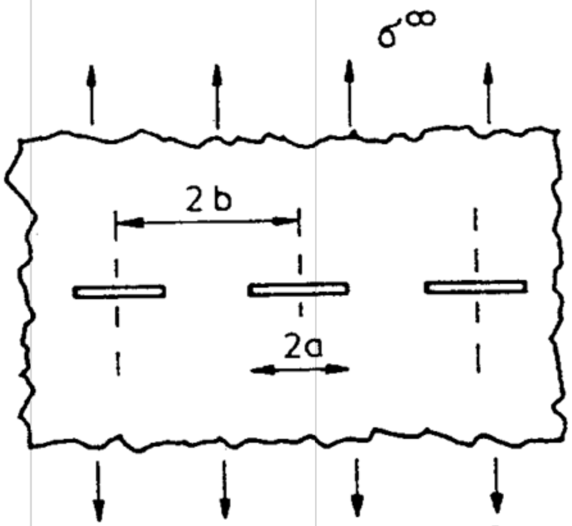
$$G = \frac{12P^2a^2}{Eb^3}$$

$$K = 2\sqrt{3}^1 P_a b^{-3/2}$$

Typical Specimen Geometries (Cont.)

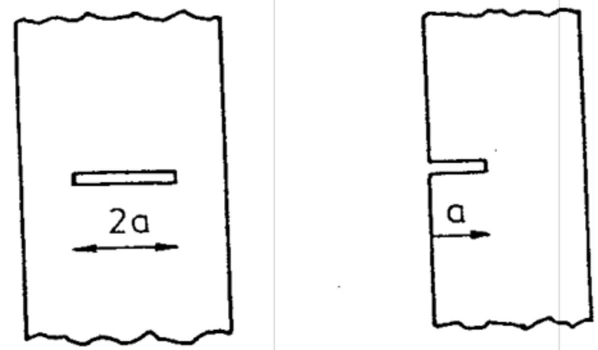
Line of cracks in tension (also Mode III)

$$K_I = \sigma^\infty \sqrt{\pi a}\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]^{\frac{1}{2}}$$



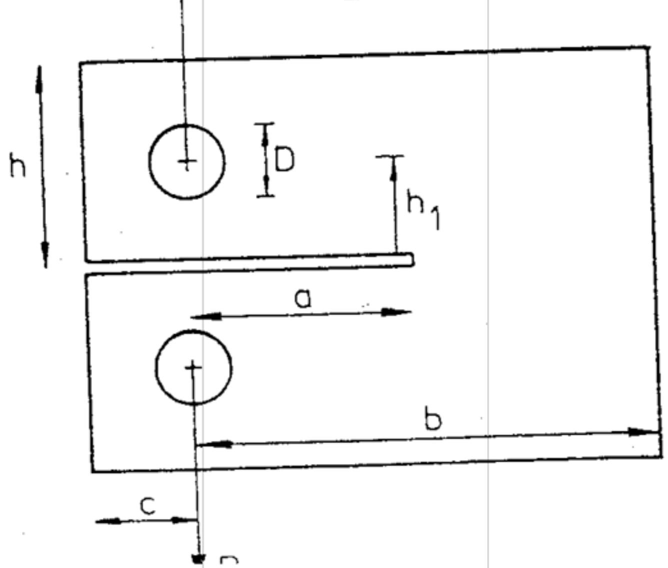
Cracks in finite width strips in Mode III

$$K_{III} = \tau^\infty \sqrt{\pi a}\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]^{\frac{1}{2}}$$



standard ASTM compact tension specimen

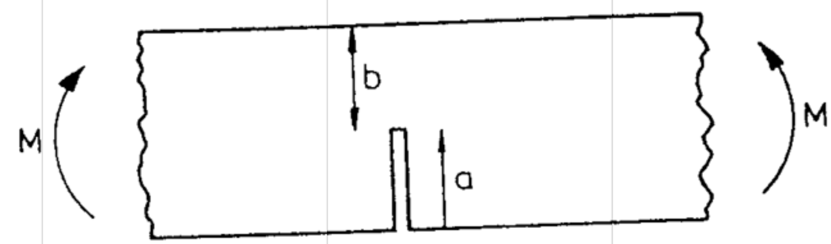
$$K_I = (P/b)\sqrt{a}\; F_1(a/b)$$



Edge crack in strip in bending

M = moment/thickness

$$K_I = Mb^{-3/2} f(a/b)$$



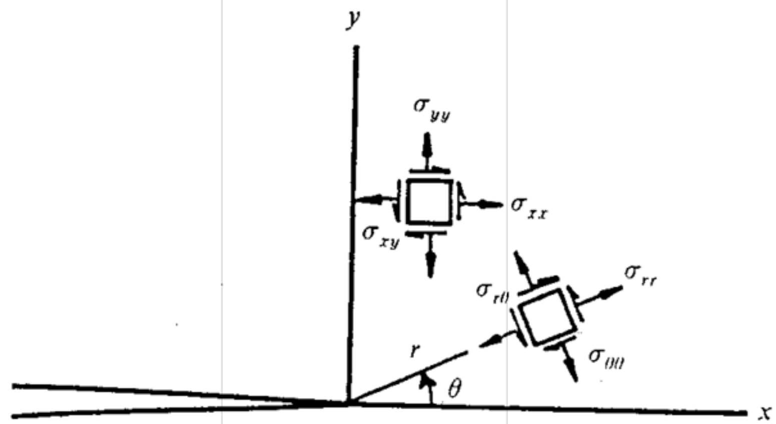
Derivation of Near-tip Fields Based on Mode I Linear Elastic Fatigue Cracks

The equilibrium equations in polar coordinates are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0,$$
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0, \qquad (1)$$

where r and θ are the polar coordinates.

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r},$$
$$\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta},$$
$$\epsilon_{r\theta} = \left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right] \times \left(\frac{1}{2}\right).$$



The condition of strain compatibility :

$$\frac{\partial^2 \epsilon_{\theta\theta}}{\partial r^2} + \frac{2}{r}\frac{\partial \epsilon_{\theta\theta}}{\partial r} - \frac{1}{r}\frac{\partial^2 \epsilon_{r\theta}}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \epsilon_{r\theta}}{\partial \theta} + \frac{1}{r^2}\frac{\partial \epsilon_{rr}^2}{\partial \theta^2} - \frac{1}{r}\frac{\partial \epsilon_{rr}}{\partial r} = 0. \qquad (2)$$

Hooke's law for plane stress $(\sigma_{zz} = 0)$,

$$E\epsilon_{rr} = \sigma_{rr} - \nu\sigma_{\theta\theta},$$
$$E\epsilon_{\theta\theta} = \sigma_{\theta\theta} - \nu\sigma_{rr},$$
$$2\mu\epsilon_{r\theta} = \mu\gamma_{r\theta} = \sigma_{r\theta},$$

where μ is the shear modulus and, for plane strain $(\epsilon_{zz} = 0)$,

$$2\mu\epsilon_{rr} = (1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta},$$
$$2\mu\epsilon_{\theta\theta} = (1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr},$$
$$2\mu\epsilon_{r\theta} = \sigma_{r\theta}.$$

The equilibrium Eqs. (1) are satisfied when the stress components are expressed by the Airy stress function χ through

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2},$$
$$\sigma_{\theta\theta} = \frac{\partial^2\chi}{\partial r^2},$$
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right).$$

The compatibility condition Eq. (2) when expressed in terms of the Airy stress function, satisfies the biharmonic equation,

$$\nabla^2\left(\nabla^2\chi\right) = 0,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

is the polar Laplacian.

The boundary conditions are:

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad \text{for} \quad \theta = \pm\pi.$$

A possible form of χ which accounts for singular stresses at the crack and is single-valued :

$$\chi = r^2 p(r,\theta) \; + \; q(r,\theta),$$

where p and q are harmonic functions of r and θ which satisfy the Laplace equations

$$\nabla^2 p = 0 \text{ and } \nabla^2 q = 0.$$

Consider solutions of separable form for the Airy stress function, $\chi = R(r)\Theta(\theta)$, **Based on**

$$p = A_1 r^{\lambda} \cos \lambda\theta \; + \; A_2 r^{\lambda} \sin \lambda\theta,$$
$$q = B_1 r^{(\lambda+2)} \cos (\lambda + 2)\theta \; + \; B_2 r^{(\lambda+2)} \sin (\lambda + 2)\theta,$$

which lead to

$$\chi = r^{(\lambda+2)} \left[A_1 \cos \lambda\theta \; + \; B_1 \cos (\lambda + 2)\theta\right] + r^{(\lambda+2)} \left[A_2 \sin \lambda\theta \; + \; B_2 \sin (\lambda + 2)\theta\right].$$

Model I fields:

$$\sigma_{\theta\theta} = \frac{\partial^2 \chi}{\partial r^2} = (\lambda + 2)\,(\lambda + 1)\, r^{\lambda} \left[A_1 \cos \lambda\theta \; + \; B_1 \cos (\lambda + 2)\theta\right],$$
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial \chi}{\partial \theta}\right) = (\lambda + 1)\, r^{\lambda} \left[\lambda A_1 \sin \lambda\theta \; + \; (\lambda + 2) B_1 \sin (\lambda + 2)\theta\right].$$

Applying the boundary conditions, obtains

$$(A_1 + B_1)\ \cos\lambda\pi = 0,$$
$$[\lambda A_1\ +\ (\lambda+2)B_1]\ \sin\lambda\pi = 0.$$

(i) cos $\lambda\pi = 0$

$$\lambda = \frac{2Z+1}{2}, \qquad\qquad B_1 = -\frac{\lambda}{\lambda+2}\,A_1,$$

where Z is an integer including zero.

(ii) sin $\lambda\pi = 0$

$$\lambda = Z \qquad \text{and} \qquad B_1 = -A_1.$$

Hence $$\lambda = \frac{Z}{2},$$

where Z is a positive or negative integer, including zero.

$$\phi = \frac{1}{2}\,\sigma_{ij}\epsilon_{ij}\ \sim\ r^{2\lambda}.$$

where ϕ **is the strain energy density.**